Homework of Ultracold Atomic Physics

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Part I

Atomic and Few-Body Physics

Chapter 1

A Single Atom

1.1

To calculate the commutator $[\boldsymbol{J}, H]$, we first calculate the following terms:

$$\begin{aligned}
\left[(p\times L)_i, p^2\right] &= [\epsilon_{kli} p_k L_l, p_j p_j] \\
&= [\epsilon_{kli}\epsilon_{mnl} p_k r_m p_n, p_j p_j] \\
&= [(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}) p_k r_m p_n, p_j p_j] \\
&= [p_k r_i p_k, p_j p_j] - [p_k r_k p_i, p_j p_j] \\
&= p_k [r_i, p_j p_j] p_k - p_k [r_k, p_j p_j] p_i \\
&= p_k (2\mathrm{i}\hbar p_i) p_k - p_k (2\mathrm{i}\hbar p_k) p_i \\
&= 0
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
\left[(L\times p)_i, p^2\right] &= [\epsilon_{kli} L_k p_l, p_j p_j] \\
&= [\epsilon_{kli}\epsilon_{mnk} r_m p_n p_l, p_j p_j] \\
&= [r_l p_i p_l, p_j p_j] - [r_i p_l p_l, p_j p_j] \\
&= (2\mathrm{i}\hbar p_l) p_i p_l - (2\mathrm{i}\hbar p_i) p_l p_l \\
&= 0
\end{aligned} \tag{1.2}$$

$$[p_i, r] = -\mathrm{i}\hbar \frac{r_i}{r} \tag{1.3}$$

$$[p_i, \frac{1}{r}] = \mathrm{i}\hbar \frac{r_i}{r^3} \tag{1.4}$$

$$\begin{aligned}\left[\frac{r_i}{r}, p^2\right] &= [r_i, p^2]\frac{1}{r} + r_i\left[\frac{1}{r}, p^2\right] \\ &= 2\mathrm{i}\hbar p_i\frac{1}{r} + r_i\left(\left[\frac{1}{r}, p_j\right]p_j + p_j\left[\frac{1}{r}, p_j\right]\right) \\ &= 2\mathrm{i}\hbar p_i\frac{1}{r} - \mathrm{i}\hbar\left(\frac{r_i r_j}{r^3}p_j + r_i p_j\frac{r_j}{r^3}\right)\end{aligned} \tag{1.5}$$

$$\begin{aligned}\left[(p\times L)_i, \frac{1}{r}\right] &= \left[p_k r_i p_k, \frac{1}{r}\right] - \left[p_k r_k p_i, \frac{1}{r}\right] \\ &= \left(\left[p_k, \frac{1}{r}\right]r_i p_k + p_k r_i\left[p_k, \frac{1}{r}\right]\right) - \left(\left[p_k, \frac{1}{r}\right]r_k p_i + p_k r_k\left[p_i, \frac{1}{r}\right]\right) \\ &= \mathrm{i}\hbar\left(\frac{r_k}{r^3}r_i p_k + p_k r_i\frac{r_k}{r^3}\right) - \mathrm{i}\hbar\left(\frac{r_k}{r^3}r_k p_i + p_k r_k\frac{r_i}{r^3}\right) \\ &= \mathrm{i}\hbar\left(\frac{r_k r_i}{r^3}p_k\right) - \mathrm{i}\hbar\frac{1}{r}p_i\end{aligned} \tag{1.6}$$

$$\begin{aligned}\left[(L\times p)_i, \frac{1}{r}\right] &= \left[r_l p_i p_l, \frac{1}{r}\right] - \left[r_i p_l p_l, \frac{1}{r}\right] \\ &= \left(r_l\left[p_i, \frac{1}{r}\right]p_l + r_l p_i\left[p_l, \frac{1}{r}\right]\right) - \left(r_i\left[p_l, \frac{1}{r}\right]p_l + r_i p_l\left[p_l, \frac{1}{r}\right]\right) \\ &= \mathrm{i}\hbar\left(r_l p_i\frac{r_l}{r^3} - \frac{r_i r_l}{r^3}p_l\right)\end{aligned} \tag{1.7}$$

$$\left[\frac{r_i}{r}, \frac{1}{r}\right] = 0 \tag{1.8}$$

Note that

$$\mathrm{i}\hbar\frac{1}{r}p_i = \mathrm{i}\hbar p_i\frac{1}{r} + \mathrm{i}\hbar\left[\frac{1}{r}, p_i\right] = \mathrm{i}\hbar\left(p_i\frac{1}{r} - \mathrm{i}\hbar\frac{r_i}{r^3}\right) \tag{1.9}$$

$$\mathrm{i}\hbar r_l p_i\frac{r_l}{r^3} = \mathrm{i}\hbar\left(p_i r_l\frac{r_l}{r^3} + \mathrm{i}\hbar\frac{r_i}{r^3}\right) = \mathrm{i}\hbar\left(p_i\frac{1}{r} + \mathrm{i}\hbar\frac{r_i}{r^3}\right) \tag{1.10}$$

Further, we can verify that $\left[p_k, \frac{r_k}{r^3}\right] = [p_k, r_k]\frac{1}{r^3} + r_k\left[p_k, \frac{1}{r^3}\right] = 0$, hence $\frac{r_k}{r^3}p_k = p_k\frac{r_k}{r^3}$.

So the commutator

$$
\begin{aligned}
[J_i, H] &= \left[\frac{1}{2m}\left(p\times L - L\times p\right)_i - Z\kappa\frac{r_i}{r}, \frac{p^2}{2m} - \frac{Z\kappa}{r}\right] \\
&= \frac{1}{(2m)^2}\left(\left[(p\times L)_i, p^2\right] - \left[(L\times p)_i, p^2\right]\right) - \frac{Z\kappa}{2m}\left[\frac{r_i}{r}, p^2\right] \\
&\quad - \frac{Z\kappa}{2m}\left(\left[(p\times L)_i, \frac{1}{r}\right] - \left[(L\times p)_i, \frac{1}{r}\right]\right) + (Z\kappa)^2\left[\frac{r_i}{r}, \frac{1}{r}\right] \\
&= -\frac{Z\kappa}{2m}\left[\frac{r_i}{r}, p^2\right] - \frac{Z\kappa}{2m}\left[(p\times L)_i, \frac{1}{r}\right] + \frac{Z\kappa}{2m}\left[(L\times p)_i, \frac{1}{r}\right] \\
&\propto \left[\frac{r_i}{r}, p^2\right] + \left[(p\times L)_i, \frac{1}{r}\right] - \left[(L\times p)_i, \frac{1}{r}\right] \\
&= 2\mathrm{i}\hbar p_i\frac{1}{r} - \mathrm{i}\hbar\left(\frac{r_i r_j}{r^3}p_j + r_i p_j\frac{r_j}{r^3}\right) \\
&\quad + \mathrm{i}\hbar\left(\frac{r_k r_i}{r^3}p_k\right) - \mathrm{i}\hbar\frac{1}{r}p_i - \mathrm{i}\hbar\left(r_l p_i\frac{r_l}{r^3} - \frac{r_i r_l}{r^3}p_l\right) \\
&= \left(2\mathrm{i}\hbar p_i\frac{1}{r} - \mathrm{i}\hbar\frac{1}{r}p_i - \mathrm{i}\hbar r_l p_i\frac{r_l}{r^3}\right) + \left(\mathrm{i}\hbar\frac{r_i r_l}{r^3}p_l - \mathrm{i}\hbar r_i p_j\frac{r_j}{r^3}\right) \\
&= 0
\end{aligned}
\tag{1.11}
$$

is always zero, meaning the Laplace-Runge-Lenz vector is conserved under Coulomb potential.

1.2

$$
H = \alpha_f \boldsymbol{S}\cdot\boldsymbol{L} = \frac{\alpha_f}{2}\left[(\boldsymbol{S}+\boldsymbol{L})^2 - \boldsymbol{S}^2 - \boldsymbol{L}^2\right] \tag{1.12}
$$

$\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L}$ can take 1/2 and 3/2, the eigenstates for this Hamiltonian is ${}^2P_{1/2}$ and ${}^2P_{3/2}$ with eigenvalue $-\alpha_f\hbar^2$ and $\alpha_f\hbar^2/2$, respectively.

1.3

For ^{87}Rb ground state manifold, $S = 1/2, L = 0, I = 3/2$, $H = B(\mu_B g_S S_z + \mu_N g_I I_z) + \alpha_{hf} S\cdot I = B_S S_z + B_I I_z + \frac{\alpha}{2}F^2 - C$.

要解$E(B)$，需要把Hamiltonian在某种表象下写出，我们选择$|S_z, I_z; S, I\rangle = |S_z, I_z\rangle$表象，其与耦合表象$|F, F_z = S_z + I_z; S, I\rangle = |F, F_z\rangle$的关系由CG系数给出，查表得到

$$\begin{aligned}
\left|\frac{1}{2}, \frac{3}{2}\right\rangle &= |2, 2\rangle \\
\left|\frac{1}{2}, \frac{1}{2}\right\rangle &= \sqrt{\frac{3}{4}}|2, 1\rangle - \sqrt{\frac{1}{4}}|1, 1\rangle \\
\left|\frac{1}{2}, -\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{2}}|2, 0\rangle - \sqrt{\frac{1}{2}}|1, 0\rangle \\
\left|\frac{1}{2}, -\frac{3}{2}\right\rangle &= \sqrt{\frac{1}{4}}|2, -1\rangle - \sqrt{\frac{3}{4}}|1, -1\rangle \\
\left|-\frac{1}{2}, \frac{3}{2}\right\rangle &= \sqrt{\frac{1}{4}}|2, 1\rangle + \sqrt{\frac{3}{4}}|1, 1\rangle \\
\left|-\frac{1}{2}, \frac{1}{2}\right\rangle &= \sqrt{\frac{1}{2}}|2, 0\rangle + \sqrt{\frac{1}{2}}|1, 0\rangle \\
\left|-\frac{1}{2}, -\frac{1}{2}\right\rangle &= \sqrt{\frac{3}{4}}|2, -1\rangle + \sqrt{\frac{1}{4}}|1, -1\rangle \\
\left|-\frac{1}{2}, -\frac{3}{2}\right\rangle &= |2, -2\rangle
\end{aligned} \tag{1.13}$$

Hamiltonian中$B_S S_z + B_I I_z$只有对角项，$\frac{\alpha}{2}F^2 - C$只在$F_z = \mathrm{Const}$的子空间有非零项。于是可以分块写出Hamiltonian：

$$F_z = 2 : \frac{1}{2}B_S + \frac{3}{2}B_I + 3\alpha - C \tag{1.14}$$

$$F_z = 1 : \begin{pmatrix} \frac{1}{2}B_S + \frac{1}{2}B_I + \frac{5}{2}\alpha - C & \frac{\sqrt{3}}{2}\alpha \\ \frac{\sqrt{3}}{2}\alpha & -\frac{1}{2}B_S + \frac{3}{2}B_I + \frac{3}{2}\alpha - C \end{pmatrix} \tag{1.15}$$

$$F_z = 0 : \begin{pmatrix} \frac{1}{2}B_S - \frac{1}{2}B_I + 2\alpha - C & \alpha \\ \alpha & -\frac{1}{2}B_S + \frac{1}{2}B_I + 2\alpha - C \end{pmatrix} \tag{1.16}$$

$$F_z = -1 : \begin{pmatrix} \frac{1}{2}B_S - \frac{3}{2}B_I + \frac{3}{2}\alpha - C & \frac{\sqrt{3}}{2}\alpha \\ \frac{\sqrt{3}}{2}\alpha & -\frac{1}{2}B_S - \frac{1}{2}B_I + \frac{5}{2}\alpha - C \end{pmatrix} \tag{1.17}$$

$$F_z = -2 : -\frac{1}{2}B_S - \frac{3}{2}B_I + 3\alpha - C \tag{1.18}$$

可以解析得到八个本征值：

$$
\begin{aligned}
&\frac{1}{2}B_S+\frac{3}{2}B_I+3\alpha-C\\
&\frac{1}{2}\sqrt{(B_S-B_I)^2+2\alpha(B_S-B_I)+4\alpha^2}+B_I+2\alpha-C\\
&\frac{1}{2}\sqrt{(B_S-B_I)^2+4\alpha^2}+2\alpha-C\\
&\frac{1}{2}\sqrt{(B_S-B_I)^2-2\alpha(B_S-B_I)+4\alpha^2}-B_I+2\alpha-C\\
&-\frac{1}{2}B_S-\frac{3}{2}B_I+3\alpha-C\\
&-\frac{1}{2}\sqrt{(B_S-B_I)^2-2\alpha(B_S-B_I)+4\alpha^2}-B_I+2\alpha-C\\
&-\frac{1}{2}\sqrt{(B_S-B_I)^2+4\alpha^2}+2\alpha-C\\
&-\frac{1}{2}\sqrt{(B_S-B_I)^2+2\alpha(B_S-B_I)+4\alpha^2}+B_I+2\alpha-C
\end{aligned}
\tag{1.19}
$$

可以看出，

1. 在外场为0时，前五个本征值为$3\alpha-C$，后三个本征值为$\alpha-C$，存在简并；
2. 在弱场时，$F_z=2,-2$的态随磁场是线性的，其余态与磁场的依赖均为二次型；
3. 在强场时，所有本征值和磁场均为线性，前四个斜率为正，后四个斜率为负，这与课本上的分析一致。

图1.1是选取一组参数画出的能级，不代表实际Rb原子的Zeeman能级。

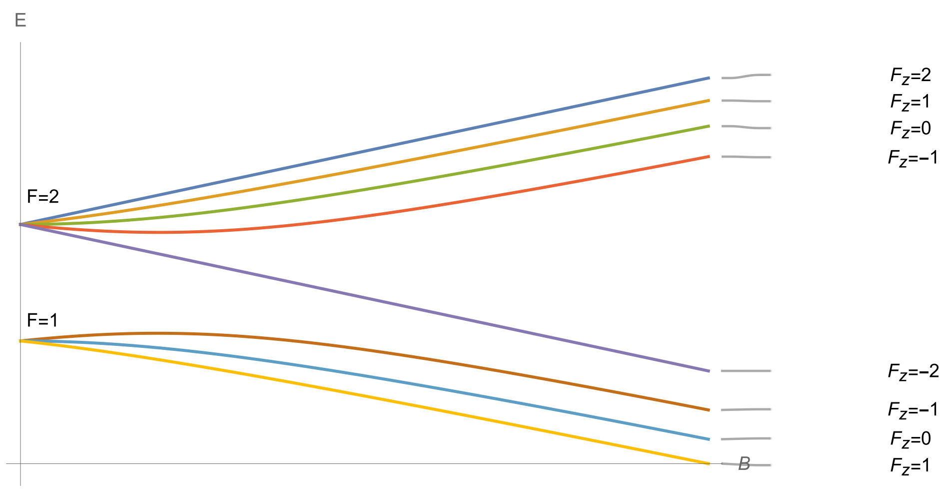

Figure 1.1: Schematic of the Zeeman energy structure.

1.8

Change into the rotating frame $U = e^{-\mathrm{i}\omega_0 t P_e} = P_g + P_e e^{-\mathrm{i}\omega_0 t}$, neglecting terms rotating at $2\omega_0$:

$$\begin{aligned} U^\dagger H U &= \left(P_g + P_e e^{\mathrm{i}\omega_0 t}\right)\left(\omega F_z + B_0 \frac{e^{\mathrm{i}\omega_0 t} + e^{-\mathrm{i}\omega_0 t}}{2} F_x\right)\left(P_g + P_e e^{-\mathrm{i}\omega_0 t}\right) \\ &= \omega F_z + \frac{B_0}{2} F_x \end{aligned} \tag{1.20}$$

$$\begin{aligned} \mathrm{i}\hbar\left(\partial_t U^\dagger\right) U &= \mathrm{i}\hbar\left(\mathrm{i}\omega_0 P_e e^{\mathrm{i}\omega_0 t}\right)\left(P_g + P_e e^{-\mathrm{i}\omega_0 t}\right) \\ &= -\hbar\omega_0 P_e \\ &= -\hbar\omega_0 \frac{\sigma_z + 1}{2} \\ &= -\omega_0 F_z - \frac{1}{2}\hbar\omega_0 \end{aligned} \tag{1.21}$$

The Schrodinger equation

$$\begin{aligned} \mathrm{i}\hbar\partial_t\left(U^\dagger\psi\right) &= \mathrm{i}\hbar U^\dagger \partial_t \psi + \mathrm{i}\hbar\left(\partial_t U^\dagger\right)\psi \\ &= U^\dagger H \psi + \mathrm{i}\hbar\left(\partial_t U^\dagger\right)\psi \\ &= U^\dagger H U (U^\dagger\psi) + \mathrm{i}\hbar\left(\partial_t U^\dagger\right) U (U^\dagger\psi) \\ &= \left((\omega - \omega_0) F_z + \frac{B_0}{2} F_x - \frac{1}{2}\hbar\omega_0\right)(U^\dagger\psi) \end{aligned} \tag{1.22}$$

the new effective Hamiltonian is

$$H_{RWA} = \Delta F_z + \frac{B_0}{2} F_x \tag{1.23}$$

where $\Delta = \omega - \omega_0$ and we throw the constant term $-\hbar\omega_0/2$.

Chapter 2

Two-Body Interaction

2.1

Write the s-wavefunction as $\Psi = u(r)/r$, the Schrodinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} - V_0 u = Eu \tag{2.1}$$

where $V_0 > 0, E \sim 0$. Solve this equation we have

$$u(r) = c\sin\left(\sqrt{\frac{2m(E+V_0)}{\hbar^2}}r\right), \quad 0 < r < r_0 \tag{2.2}$$

By definition, the scattering length

$$a_s = -\frac{\tan\left(\sqrt{\frac{2mV_0}{\hbar^2}}r_0\right)}{\sqrt{\frac{2mV_0}{\hbar^2}}} \tag{2.3}$$

For a bound state, $E = -E_0 < 0$, thus

$$u(r) = c' e^{-\sqrt{\frac{2mE_0}{\hbar^2}}r}, \quad r > r_0 \tag{2.4}$$

Both $u(r)$ and the derivative of $u(r)$ is continuous at r_0, the binding energy is given by

$$-\frac{1}{a_s} = \frac{\sqrt{\frac{2m(E_0+V_0)}{\hbar^2}}}{\tan\left(\sqrt{\frac{2m(E_0+V_0)}{\hbar^2}}r_0\right)} = -\sqrt{\frac{2mE_0}{\hbar^2}} \Rightarrow E = -E_0 = -\frac{\hbar^2}{2ma_s^2} \tag{2.5}$$

2.2

Like 2.1, the radial wavefunction here is

$$u(r) = c\sinh\left(\sqrt{\frac{2mV_0}{\hbar^2}}r\right), \quad 0 < r < r_0 \tag{2.6}$$

By definition, the scattering length

$$a_s = -\frac{\tanh\left(\sqrt{\frac{2mV_0}{\hbar^2}}r_0\right)}{\sqrt{\frac{2mV_0}{\hbar^2}}} \tag{2.7}$$

The tangent function in 2.1 becomes hyperbolic tangent function, thus the scattering length will not change repeatedly and is bounded by $(-\sqrt{\frac{\hbar^2}{2mV_0}}, 0)$.

2.3

对l-wave，经过势场$U(r)$散射后的径向波函数$u(r)$在$r \to +\infty$的渐进行为是

$$u(r)|_{r\to+\infty} \sim \frac{1}{k}\sin(kr - l\pi/2 + \delta_{kl}) \tag{2.8}$$

沿z方向的自由粒子径向波函数$v(r)$在$r \to +\infty$的渐进行为是

$$v(r)|_{r\to+\infty} \sim \frac{1}{k}\sin(kr - l\pi/2) \tag{2.9}$$

由Schrodinger方程有$\frac{\mathrm{d}}{\mathrm{d}r}\left(u\frac{\mathrm{d}v}{\mathrm{d}r} - v\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -\frac{2m}{\hbar^2}uvU(r)$，将上面的波函数带入，有

$$u\frac{\mathrm{d}v}{\mathrm{d}r} - v\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{k}\sin\delta_{kl} \tag{2.10}$$

如果把势场当做微扰，可以将u近似为v；如果势场是short-range，且入射粒子能量很低，v可以近似写作$r\frac{(kr)^l}{(2l+1)!!}$。带入以上近似，积分得到

$$\begin{aligned}
\sin\delta_{kl} &= -\frac{2mk}{\hbar^2}\int_0^{+\infty} uvU(r)\mathrm{d}r \\
&\approx -\frac{2mk}{\hbar^2}\int_0^{+\infty} r^2\frac{(kr)^{2l}}{[(2l+1)!!]^2}U(r)\mathrm{d}r \\
&= -\frac{2mk^{2l+1}}{[(2l+1)!!]^2\hbar^2}\int_0^{+\infty} r^{2(l+1)}U(r)\mathrm{d}r \\
&\propto k^{2l+1}
\end{aligned} \tag{2.11}$$

Ref: Prof. Huanxiong Yang's slides on QM.

Part II

Interacting Bose Gas

Chapter 3

Interaction Effects

3.1

If there is a Bose–Einstein condensation (BEC), the chemical potential mu should be zero. In 1D case,

$$\int_0^\infty 2\mathrm{d}k \frac{1}{e^{\hbar^2 k^2/2mk_b T} - 1} \propto \int_0^\infty \mathrm{d}z \frac{1}{e^{z^2} - 1} \propto \int_0^\infty \mathrm{d}z \frac{1}{\sqrt{z}(e^z - 1)}$$

doesn't converge. Similarly, in 2D case,

$$\int_0^\infty 2\pi k \mathrm{d}k \frac{1}{e^{\hbar^2 k^2/2mk_b T} - 1} \propto \int_0^\infty \mathrm{d}z \frac{1}{e^z - 1}$$

doesn't converge either. Thus, no BEC in 1D and 2D. As a contrast, the integration in 3D $\propto \int_0^\infty \mathrm{d}z \frac{\sqrt{z}}{e^z - 1}$ converges.

3.2

Above T_c:

$$\begin{aligned}
\rho(\boldsymbol{r}, \boldsymbol{r'}) &= \langle r| e^{-H/k_B T} |r'\rangle = \sum_{k,k'} \langle r|k\rangle \langle k| e^{-H/k_B T} |k'\rangle \langle k'|r'\rangle = \sum_k e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} e^{-\hbar^2 k^2/2mk_B T} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r'}} \\
&= \frac{V}{(2\pi)^3} \iiint k^2 \mathrm{d}k \sin\theta \mathrm{d}\theta \mathrm{d}\varphi e^{-\hbar^2 k^2/2mk_B T} e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r'})} \\
&= \frac{V}{(2\pi)^3} \iiint k^2 \mathrm{d}k \sin\theta \mathrm{d}\theta \mathrm{d}\varphi e^{-\hbar^2 k^2/2mk_B T} e^{-\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r'}|\cos\theta} \\
&= \frac{V}{(2\pi)^3} \int k^2 e^{-\hbar^2 k^2/2mk_B T} \frac{4\pi \sin k|\boldsymbol{r}-\boldsymbol{r'}|}{k|\boldsymbol{r}-\boldsymbol{r'}|} \mathrm{d}k \\
&= \frac{1}{(2\pi)^{3/2}} \frac{V}{\lambda^3} \exp\left(-\frac{|\boldsymbol{r}-\boldsymbol{r'}|^2}{2\lambda^2}\right)
\end{aligned} \tag{3.1}$$

where $\lambda = \sqrt{\frac{\hbar^2}{mk_BT}}$. This indicates no ODLRO above T_c.

Below T_c, by definition we have

$$\begin{aligned}\rho(\boldsymbol{r}, \boldsymbol{r'}) &= N_0\psi^*(\boldsymbol{r})\psi(\boldsymbol{r'}) \\ &= N_0/V \\ &\approx n\end{aligned} \tag{3.2}$$

this means the existence of ODLRO.

3.3

Denote the unitary transformation

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{3.3}$$

and

$$\begin{pmatrix} \alpha_k \\ \alpha^\dagger_{-k} \end{pmatrix} = U \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} \tag{3.4}$$

where $a_k^{(\dagger)}$ is bosonic operator that satisfies The commutative relation requires

$$\begin{aligned}1 &= [\alpha_k, \alpha_k^\dagger] = u_{11}^*u_{11} - u_{12}^*u_{12} \\ 1 &= [\alpha_{-k}, \alpha_{-k}^\dagger] = u_{22}^*u_{22} - u_{21}^*u_{21}\end{aligned} \tag{3.5}$$

U is unitary means that $UU^\dagger = \mathbb{1}$:

$$\begin{aligned}u_{11}^*u_{11} + u_{12}^*u_{12} = 1 \\ u_{21}^*u_{21} + u_{22}^*u_{22} = 1\end{aligned} \tag{3.6}$$

Solve these equations we have

$$U = \begin{pmatrix} e^{\mathrm{i}\theta} & 0 \\ 0 & e^{\mathrm{i}\varphi} \end{pmatrix} \tag{3.7}$$

This is a trivial transformation which only gives a phase.

3.4

The ground state is the vacuum of quasi-particle:

$$\begin{aligned}\alpha_k |G\rangle &= (u_k a_k + v_k a_{-k}^\dagger) e^{\sqrt{N_0} a_{k=0}^\dagger} e^{-\sum_{k\neq 0} \frac{v_k}{u_k} a_k^\dagger a_{-k}^\dagger} |0\rangle \\ &= e^{\sqrt{N_0} a_{k=0}^\dagger} (u_k a_k + v_k a_{-k}^\dagger) e^{-\sum_{k\neq 0} \frac{v_k}{u_k} a_k^\dagger a_{-k}^\dagger} |0\rangle \\ &= e^{\sqrt{N_0} a_{k=0}^\dagger} \left[u_k \left(-\frac{v_k}{u_k} a_{-k}^\dagger \right) + v_k a_{-k}^\dagger \right] e^{-\sum_{k\neq 0} \frac{v_k}{u_k} a_k^\dagger a_{-k}^\dagger} |0\rangle \\ &= 0\end{aligned} \tag{3.8}$$

$$\begin{aligned}\langle G| a_{k=0} |G\rangle &= \langle G| a_k e^{\sqrt{N_0} a_{k=0}^\dagger} e^{-\sum_{k\neq 0} \frac{v_k}{u_k} a_k^\dagger a_{-k}^\dagger} |0\rangle \\ &= \langle G| e^{-\sum_{k\neq 0} \frac{v_k}{u_k} a_k^\dagger a_{-k}^\dagger} a_k e^{\sqrt{N_0} a_{k=0}^\dagger} |0\rangle \\ &= \langle G| e^{-\sum_{k\neq 0} \frac{v_k}{u_k} a_k^\dagger a_{-k}^\dagger} \sqrt{N_0} e^{\sqrt{N_0} a_{k=0}^\dagger} |0\rangle \\ &= \langle G| \sqrt{N_0} |G\rangle \\ &= \sqrt{N_0}\end{aligned} \tag{3.9}$$

3.5

For bosons:

$$\begin{aligned}\text{LHS} &= \frac{e^{\beta\varepsilon_{\boldsymbol{k}}}}{e^{\beta\varepsilon_{\boldsymbol{k}}}-1} \frac{e^{\beta\varepsilon_{\boldsymbol{q}}}}{e^{\beta\varepsilon_{\boldsymbol{q}}}-1} \frac{1}{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}-1} \\ &= \frac{1}{e^{\beta\varepsilon_{\boldsymbol{k}}}-1} \frac{1}{e^{\beta\varepsilon_{\boldsymbol{q}}}-1} \frac{e^{\beta(\varepsilon_{\boldsymbol{k}}+\varepsilon_{\boldsymbol{q}})}}{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}-1} \\ &= \frac{1}{e^{\beta\varepsilon_{\boldsymbol{k}}}-1} \frac{1}{e^{\beta\varepsilon_{\boldsymbol{q}}}-1} \frac{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}}{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}-1} \\ &= \text{RHS}\end{aligned} \tag{3.10}$$

For fermions:

$$\begin{aligned}\text{LHS} &= \frac{e^{\beta\varepsilon_{\boldsymbol{k}}}}{e^{\beta\varepsilon_{\boldsymbol{k}}}+1} \frac{e^{\beta\varepsilon_{\boldsymbol{q}}}}{e^{\beta\varepsilon_{\boldsymbol{q}}}+1} \frac{1}{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}+1} \\ &= \frac{1}{e^{\beta\varepsilon_{\boldsymbol{k}}}+1} \frac{1}{e^{\beta\varepsilon_{\boldsymbol{q}}}+1} \frac{e^{\beta(\varepsilon_{\boldsymbol{k}}+\varepsilon_{\boldsymbol{q}})}}{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}+1} \\ &= \frac{1}{e^{\beta\varepsilon_{\boldsymbol{k}}}+1} \frac{1}{e^{\beta\varepsilon_{\boldsymbol{q}}}+1} \frac{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}}{e^{\beta\varepsilon_{\boldsymbol{k}+\boldsymbol{q}}}+1} \\ &= \text{RHS}\end{aligned} \tag{3.11}$$

3.9

Note that the Hamiltonian matrix is tri-diagonal:

$$\begin{aligned} H_{ii} &= \langle i, N-i| H |i, N-i\rangle = \frac{U}{2}[i(i-1)+(N-i)(N-i-1)] \\ H_{i,i+1} &= \langle i, N-i| H |i+1, N-i-1\rangle = -J\sqrt{(i+1)(N-i)} \\ H_{i,i-1} &= \langle i, N-i| H |i-1, N-i+1\rangle = -J\sqrt{i(N-i+1)} \end{aligned} \tag{3.12}$$

Numerically diagonalize the matrix with total particle number $N = 1000$ we have the ground state wave function, relative particle fluctuation $(\Delta N)^2(\Delta N = N_1 - N_2)$ and the energy gap in 4 regimes.

1. $U > 0, U \gg J$:

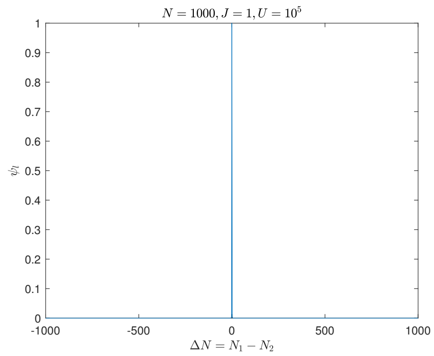

Figure 3.1: Wave function with $J = 1, U = 10^5$. $\langle(\Delta N)^2\rangle = 2.0037 \times 10^{-4} \to 0$.

2. $U > 0, U \lesssim J$:

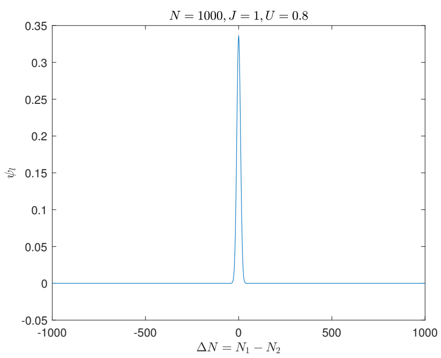

Figure 3.2: Wave function with $J = 1, U = 0.8$. $\langle(\Delta N)^2\rangle = 49.7117$.

3. $U > 0, U \ll J$:

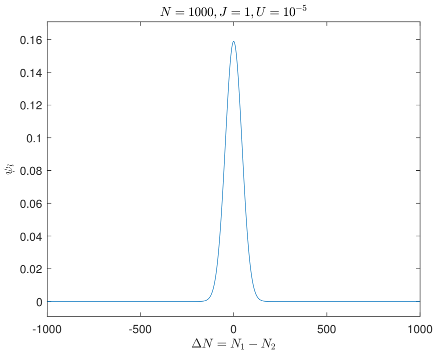

Figure 3.3: Wave function with $J = 1, U = 10^{-5}$. $\langle(\Delta N)^2\rangle = 997.5118 \approx N$.

4. $U < 0, |U| \gg J$[1]:

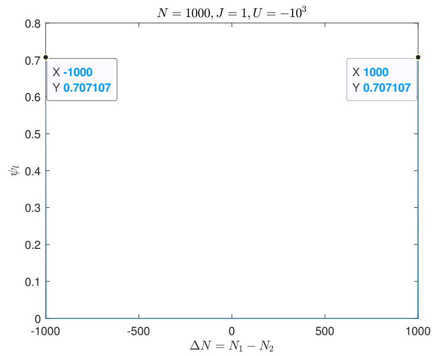

Figure 3.4: Wave function with $J = 1, U = -10^3$. $\langle(\Delta N)^2\rangle = 10^6$.

In the fourth regime, we calculate the gap between the first excited state and the ground state as a function of total particle number, see Fig. 3.5.

3.10

Assume $\psi(\boldsymbol{r}) = a_1\phi_1(\boldsymbol{r}) + a_2\phi_2(\boldsymbol{r})$.

For bosons,

$$\begin{aligned} \rho(\boldsymbol{r}) &= \langle\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\rangle \\ &= \langle a_1^\dagger a_1\rangle|\phi_1(\boldsymbol{r})|^2 + \langle a_2^\dagger a_2\rangle|\phi_2(\boldsymbol{r})|^2 \\ &= n_1|\phi_1(\boldsymbol{r})|^2 + n_2|\phi_2(\boldsymbol{r})|^2 \end{aligned} \tag{3.13}$$

[1]Note that there's degeneracy in ground state when $U < 0$. Here we take a symmetric wave function as ground state wave function.

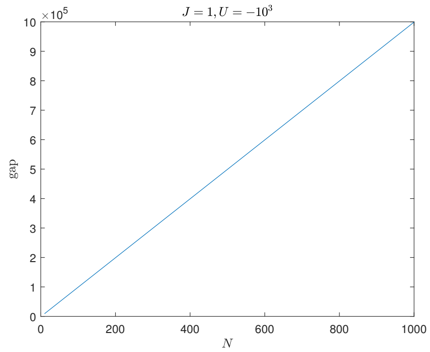

Figure 3.5: Energy gap as a function of total particle number N under $U < 0, |U| \gg J$.

$$\begin{aligned}\langle\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\psi^\dagger(\boldsymbol{r'})\psi(\boldsymbol{r'})\rangle =& n_1^2|\phi_1(\boldsymbol{r})|^2|\phi_1(\boldsymbol{r'})|^2 + n_2|\phi_2(\boldsymbol{r})|^2|\phi_2(\boldsymbol{r'})|^2\\ &+n_1n_2|\phi_1(\boldsymbol{r})|^2|\phi_2(\boldsymbol{r'})|^2 + n_1n_2|\phi_1(\boldsymbol{r'})|^2|\phi_2(\boldsymbol{r})|^2\\ &+n_1n_2\left[\phi_1^*(\boldsymbol{r})\phi_1(\boldsymbol{r'})\phi_2(\boldsymbol{r})\phi_2^*(\boldsymbol{r'}) + h.c.\right]\\ =&\rho(\boldsymbol{r})\rho(\boldsymbol{r'})\\ &+n_1(1+n_2)\phi_1^*(\boldsymbol{r})\phi_1(\boldsymbol{r'})\phi_2(\boldsymbol{r})\phi_2^*(\boldsymbol{r'}) + n_2(1+n_1)\phi_1(\boldsymbol{r})\phi_1^*(\boldsymbol{r'})\phi_2^*(\boldsymbol{r})\phi_2(\boldsymbol{r'})\end{aligned} \tag{3.14}$$

For fermions,

$$\begin{aligned}\langle\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\psi^\dagger(\boldsymbol{r'})\psi(\boldsymbol{r'})\rangle =& n_1^2|\phi_1(\boldsymbol{r})|^2|\phi_1(\boldsymbol{r'})|^2 + n_2|\phi_2(\boldsymbol{r})|^2|\phi_2(\boldsymbol{r'})|^2\\ &+n_1n_2|\phi_1(\boldsymbol{r})|^2|\phi_2(\boldsymbol{r'})|^2 + n_1n_2|\phi_1(\boldsymbol{r'})|^2|\phi_2(\boldsymbol{r})|^2\\ &+\langle a_1^\dagger a_2 a_2^\dagger a_1\rangle\phi_1^*(\boldsymbol{r})\phi_1(\boldsymbol{r'})\phi_2(\boldsymbol{r})\phi_2^*(\boldsymbol{r'})\\ &+\langle a_2^\dagger a_1 a_1^\dagger a_2\rangle\phi_1(\boldsymbol{r})\phi_1^*(\boldsymbol{r'})\phi_2^*(\boldsymbol{r})\phi_2(\boldsymbol{r'})\\ =&\rho(\boldsymbol{r})\rho(\boldsymbol{r'})\\ &+n_1(1-n_2)\phi_1^*(\boldsymbol{r})\phi_1(\boldsymbol{r'})\phi_2(\boldsymbol{r})\phi_2^*(\boldsymbol{r'}) + n_2(1-n_1)\phi_1(\boldsymbol{r})\phi_1^*(\boldsymbol{r'})\phi_2^*(\boldsymbol{r})\phi_2(\boldsymbol{r'})\end{aligned} \tag{3.15}$$

The correlation terms are different.

Part III

Degenerate Fermi Gases

Chapter 4

The Fermi Superfluid

6.1

Eq. 6.8:

$$\frac{m}{4\pi\hbar^2 a_s} = \frac{1}{V}\left[\sum_{|\boldsymbol{k}|>k_{\mathrm{F}}} \frac{1}{E-2\left(\epsilon_{\boldsymbol{k}}-\mu\right)} + \sum_{\boldsymbol{k}} \frac{1}{2\epsilon_{\boldsymbol{k}}}\right] \tag{4.1}$$

Substitute sum by integration: $\sum \to \frac{V}{(2\pi)^3}\int \mathrm{d}^3k$, $\sum_{\boldsymbol{k}} = \sum_{|\boldsymbol{k}|>k_{\mathrm{F}}} + \sum_{|\boldsymbol{k}|<k_{\mathrm{F}}}$:

$$\begin{aligned} \mathrm{RHS} &= \frac{1}{V}\left[\sum_{|\boldsymbol{k}|>k_{\mathrm{F}}}\left(\frac{1}{E-2\left(\epsilon_{\boldsymbol{k}}-\mu\right)} + \frac{1}{2\epsilon_{\boldsymbol{k}}}\right) + \sum_{|\boldsymbol{k}|<k_{\mathrm{F}}} \frac{1}{2\epsilon_{\boldsymbol{k}}}\right] \\ &= \frac{1}{V}\left[\sum_{|\boldsymbol{k}|>k_{\mathrm{F}}} \frac{E+2\mu}{2\epsilon_{\boldsymbol{k}}(E+2\mu-2\epsilon_{\boldsymbol{k}})} + \sum_{|\boldsymbol{k}|<k_{\mathrm{F}}} \frac{1}{2\epsilon_{\boldsymbol{k}}}\right] \end{aligned} \tag{4.2}$$

Note that $\epsilon_{\boldsymbol{k}} = \frac{\hbar^2k^2}{2m}$, $\mu = \frac{\hbar^2k_{\mathrm{F}}^2}{2m}$ set $\hbar^2k_{\mathrm{F}}^2/m$ as unit energy, $c = E/(\hbar^2k_{\mathrm{F}}^2/m)$, substitute $k = xk_{\mathrm{F}}$ and we have

$$\begin{aligned} \mathrm{RHS} &= \frac{1}{(2\pi)^3}\int_{k_{\mathrm{F}}}^{\infty} \mathrm{d}^3k \frac{E+2\mu}{2\epsilon_{\boldsymbol{k}}(E+2\mu-2\epsilon_{\boldsymbol{k}})} + \frac{1}{(2\pi)^3}\int_{k_{\mathrm{F}}}^{\infty} \mathrm{d}^3k \frac{1}{2\epsilon_{\boldsymbol{k}}} \\ &= \frac{1}{(2\pi)^3}\int_1^{\infty} \frac{c+1}{x^2(c+1-x^2)} \frac{4\pi m k_{\mathrm{F}}}{\hbar^2} x^2 \mathrm{d}x + \frac{1}{(2\pi)^3}\frac{4\pi m k_{\mathrm{F}}}{\hbar^2} \\ &= \frac{mk_{\mathrm{F}}}{2\pi^2\hbar^2}\left[\int_1^{\infty} \frac{c+1}{c+1-x^2}\mathrm{d}x + 1\right] \end{aligned} \tag{4.3}$$

thus we have

$$\frac{1}{k_{\mathrm{F}}a_s} = \frac{2}{\pi}[I(c)+1] = f\left(\frac{E}{E_{\mathrm{F}}}\right) \tag{4.4}$$

where

$$I(c) = \int_1^{\infty} \frac{c+1}{c+1-x^2}\mathrm{d}x = \sqrt{|c+1|}\arctan\sqrt{|c+1|}, \quad c<0 \tag{4.5}$$

Take the limit $k_{\mathrm{F}} \to 0, c \to -\infty, ck_{\mathrm{F}}^2 = \frac{m}{\hbar^2}E$ we go back to two-body problem in vacuum:

$$\begin{aligned}\frac{m}{4\pi\hbar^2 a_s} &= \frac{mk_{\mathrm{F}}}{2\pi^2\hbar^2}\left[\sqrt{|c+1|}\arctan\sqrt{|c+1|}+1\right] \\ &= \frac{m}{2\pi^2\hbar^2}\sqrt{|c|k_{\mathrm{F}}^2}\frac{\pi}{2} \\ &= \frac{m}{4\pi\hbar^2}\sqrt{\frac{m}{\hbar^2}|E|} \Rightarrow E = -\frac{\hbar^2}{ma_s^2}\end{aligned} \tag{4.6}$$

which is in agreement of the energy of shallow bound state ($E = -\frac{\hbar^2}{2ma_s^2}$ for a single particle) discussed in chapter 2.

In general, the solution of Eq. 4.4 can be obtained by looking at the intersection of a constant $\frac{1}{k_{\mathrm{F}}a_s}$ with the f function.

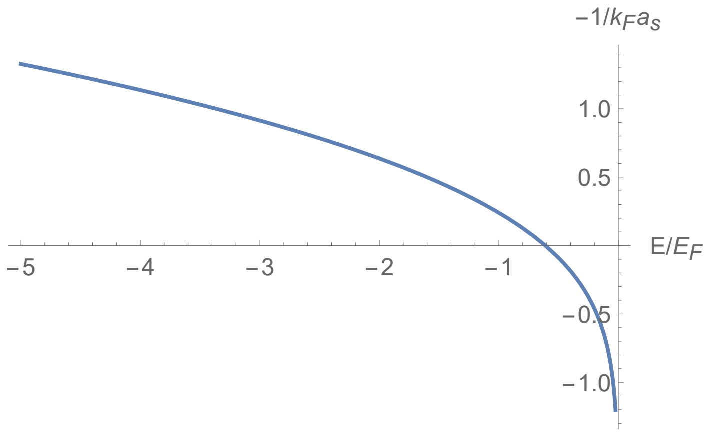

Figure 4.1: Solution of two-body problem.

6.3

This is still a quadratic Hamiltonian and can be diagonalized:

$$\begin{aligned}H_{\mathrm{BCS}} &= \begin{pmatrix} c_{\boldsymbol{k}\uparrow}^\dagger & c_{-\boldsymbol{k}\downarrow}\end{pmatrix}\begin{pmatrix}\varepsilon_{\boldsymbol{k}}-\mu_\uparrow & \Delta \\ \Delta^* & -(\varepsilon_{\boldsymbol{k}}-\mu_\downarrow)\end{pmatrix}\begin{pmatrix}c_{\boldsymbol{k}\uparrow} \\ c_{-\boldsymbol{k}\downarrow}^\dagger\end{pmatrix} + (\varepsilon_{\boldsymbol{k}}-\mu_\downarrow) \\ &= \begin{pmatrix} c_{\boldsymbol{k}\uparrow}^\dagger & c_{-\boldsymbol{k}\downarrow}\end{pmatrix}\begin{pmatrix}\varepsilon_{\boldsymbol{k}}-\mu-h/2 & \Delta \\ \Delta^* & -(\varepsilon_{\boldsymbol{k}}-\mu+h/2)\end{pmatrix}\begin{pmatrix}c_{\boldsymbol{k}\uparrow} \\ c_{-\boldsymbol{k}\downarrow}^\dagger\end{pmatrix} + (\varepsilon_{\boldsymbol{k}}-\mu_\downarrow) \\ &= \sum_{\boldsymbol{k}}\left(-\frac{h}{2}+\sqrt{(\varepsilon_{\boldsymbol{k}}-\mu)^2+|\Delta|^2}\right)\alpha_{\boldsymbol{k}}^\dagger\alpha_{\boldsymbol{k}} + \left(-\frac{h}{2}-\sqrt{(\varepsilon_{\boldsymbol{k}}-\mu)^2+|\Delta|^2}\right)\beta_{\boldsymbol{k}}\beta_{\boldsymbol{k}}^\dagger + (\varepsilon_{\boldsymbol{k}}-\mu_\downarrow) \\ &= \sum_{\boldsymbol{k}}\left(\mathcal{E}_{\boldsymbol{k}}-\frac{h}{2}\right)\alpha_{\boldsymbol{k}}^\dagger\alpha_{\boldsymbol{k}} + \left(\mathcal{E}_{\boldsymbol{k}}+\frac{h}{2}\right)\beta_{\boldsymbol{k}}^\dagger\beta_{\boldsymbol{k}} - (\mathcal{E}_{\boldsymbol{k}}-(\varepsilon_{\boldsymbol{k}}-\mu))\end{aligned} \tag{4.7}$$

where $\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2}$, $h = \mu_\uparrow - \mu_\downarrow$, $\mathcal{E}_{\boldsymbol{k}} = \sqrt{(\varepsilon_{\boldsymbol{k}} - \mu)^2 + |\Delta|^2}$, α and β are two quasi-particles. It is clear that the energy of quasi-particles is shifted $\pm h/2$ compared with free Fermi gas.

Part IV

Optical Lattices

Chapter 5

Noninteracting Bands

7.1

In small V_x regime, use the nearly free electron model $\psi = e^{\pm \mathrm{i} k_0 x}$ and the lattice potential $V_x \cos k_0 x$ can be treat as perturbation:

$$\begin{pmatrix} 0 & \frac{V_x}{4} \\ \frac{V_x}{4} & 0 \end{pmatrix} \tag{5.1}$$

So the band gap is $V_x/2$.

In large V_x regime, the lattice potential becomes deep enough and we can expand the lattice potential around the bottom of each minimum x_i. Up to the quadratic order we obtain a harmonic potential

$$H = \frac{\hbar^2 \partial_x^2}{2m} + V_x k_0^2 \left(x - x_i\right)^2 \tag{5.2}$$

Hence the band gap is

$$\hbar\omega = \hbar \sqrt{\frac{2V_x}{m}} k_0 \tag{5.3}$$

7.2

Consider a 2D square lattice with lattice potential $V_0 \left(\cos k_0 x + \cos k_0 y\right)$. Use degenerate perturbation theory we know there is a gap $2V_0$ (V_0) at $(\frac{\pi}{a}, \frac{\pi}{a})$ ($(\frac{\pi}{a}, 0)$). In nearly free electron model, the energy at $(\frac{\pi}{a}, \frac{\pi}{a})$ and $(\frac{\pi}{a}, 0)$ is $\frac{\hbar^2\pi^2}{ma^2}$ and $\frac{\hbar^2\pi^2}{2ma^2}$, respectively. To open a real band gap, V_0 should satisfy

$$\frac{3}{2} V_0 > \frac{\hbar^2 \pi^2}{2ma^2} \quad \Rightarrow \quad V_0 > \frac{\hbar^2 \pi^2}{3ma^2} \tag{5.4}$$

7.5

After adding J_3 term, the Hamiltonian has still no diagonal term:

$$H(\boldsymbol{k}) = \begin{pmatrix} 0 & \begin{array}{c} -J_1\left(1+e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_3}+e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_2}\right) \\ -J_3\left(e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_1}+e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_1}+e^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{a}_2-\boldsymbol{a}_3)}\right) \end{array} \\ \begin{array}{c} -J_1\left(1+e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_3}+e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_2}\right) \\ -J_3\left(e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_1}+e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_1}+e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{a}_2-\boldsymbol{a}_3)}\right) \end{array} & 0 \end{pmatrix} \tag{5.5}$$

Thus, without any further calculation we know two Dirac points never merge.

7.6

We numerically calculate the Chern number and plot the phase diagram.

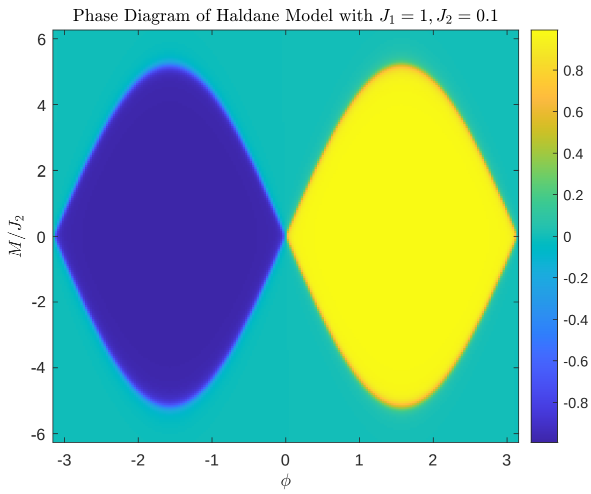

Figure 5.1: The phase diagram for the Haldane model.

Chapter 6

The Hubbard Model

8.1

The density fluctuation at each site obeys binomial distribution:

$$\begin{aligned}
P(n_i) &= C_N^{n_i}\left(\frac{1}{N_s}\right)^{n_i}\left(1-\frac{1}{N_s}\right)^{N-n_i} \\
&= \frac{N!}{N_i!(N-n_i)!}\left(\frac{1}{N_s}\right)^{n_i}\left(1-\frac{1}{N_s}\right)^{\bar{n}N_s-n_i} \\
&= \frac{N(N-1)\ldots(N-n_i+1)}{n_i!}\left(\frac{1}{N_s}\right)^{n_i}\left(1-\frac{1}{N_s}\right)^{\bar{n}N_s}\left(1-\frac{1}{N_s}\right)^{-n_i} \\
&= \frac{N^{n_i}}{n_i!}\left(\frac{1}{N_s}\right)^{n_i} e^{\bar{n}} \\
&= e^{\bar{n}}\frac{\bar{n}^{n_i}}{n_i!}
\end{aligned} \tag{6.1}$$

which can be approximated by Poisson distribution with $\bar{n} = N/N_s = \langle n_i \rangle$ when $N, N_s \gg 1$. In quantum case, we can easily verify that

$$\begin{aligned}
\langle n_i \rangle &= \langle \mathrm{SF}| b_i^\dagger b_i |\mathrm{SF}\rangle \\
&= \frac{1}{N!}\langle 0| b_{\boldsymbol{k}=0}^N b_i^\dagger b_i b_{\boldsymbol{k}=0}^{\dagger N} |0\rangle \\
&= \frac{1}{N!}\langle 0| b_{\boldsymbol{k}=0}^N \frac{1}{N_s} b_{\boldsymbol{k}=0}^\dagger b_{\boldsymbol{k}=0} b_{\boldsymbol{k}=0}^{\dagger N} |0\rangle \\
&= \frac{N}{N_s}
\end{aligned} \tag{6.2}$$

8.3

The mean-field Hamiltonian

$$\begin{aligned} H_{\mathrm{MF}} &= -\phi b^\dagger - \phi^* b + \frac{U}{2} n(n-1) - \mu n + \frac{|\phi|^2}{ZJ} \\ &= H + H_0 \end{aligned} \tag{6.3}$$

where $H = -\phi b^\dagger - \phi^* b$ is the perturbation. The perturbated energy is

$$\begin{aligned} & - \frac{\langle n_0 | H | n_0 + 1 \rangle \langle n_0 + 1 | H | n_0 \rangle}{[\frac{U}{2}(n_0+1)n_0 - mu(n_0+1)] - [\frac{U}{2} n_0 (n_0 - 1) - \mu n_0]} \\ & - \frac{\langle n_0 | H | n_0 - 1 \rangle \langle n_0 - 1 | H | n_0 \rangle}{[\frac{U}{2}(n_0-1)(n_0-2) - \mu(n_0-1)] - [\frac{U}{2} n_0 (n_0 - 1) - \mu n_0]} \\ = & - \frac{(n_0+1)|\phi|^2}{U n_0 - \mu} - \frac{n_0 |\phi|^2}{-U(n_0-1) + \mu} \end{aligned} \tag{6.4}$$

Total energy

$$\begin{aligned} E &= \frac{U}{2} n_0 (n_0+1) - \mu n_0 + \frac{|\phi|^2}{ZJ} - \frac{(n_0+1)|\phi|^2}{U n_0 - \mu} - \frac{n_0 |\phi|^2}{-U(n_0-1)+\mu} \\ &= \frac{U}{2} n_0 (n_0+1) - \mu n_0 + \frac{|\phi|^2}{ZJ} + a|\phi|^2 \end{aligned} \tag{6.5}$$

When $a = 0$, phase transition occurs, so the critical value

$$\frac{J_c}{U} = \frac{\left(n_0 - \frac{\mu}{U}\right)\left(\frac{\mu}{U} - (n_0 - 1)\right)}{Z\left(\frac{\mu}{U} + 1\right)} \tag{6.6}$$

8.4

The relativistic nonlinear equation is (there should be a '−' before ∂_t^2 term)

$$-\frac{\hbar^2 \partial^2 \phi}{\partial t^2} = -\frac{\hbar^2 \nabla^2}{2m} \phi + U|\phi|^2 \phi \tag{6.7}$$

Set $\phi = \sqrt{\rho} e^{\mathrm{i}\theta}$, we have

$$\begin{aligned} \partial_t^2 \phi = \partial_t (\partial_t \sqrt{\rho} e^{\mathrm{i}\theta}) &= \partial_t \left(\frac{1}{2\sqrt{\rho}} \dot{\rho} e^{\mathrm{i}\theta} + \mathrm{i} \sqrt{\rho} e^{\mathrm{i}\theta} \dot{\theta} \right) \\ &= -\frac{1}{4\rho\sqrt{\rho}} \dot{\rho}^2 + \frac{1}{2\sqrt{\rho}} \ddot{\rho} + \mathrm{i} \frac{1}{\sqrt{\rho}} \dot{\rho} \dot{\theta} - \sqrt{\rho} \dot{\theta}^2 + \mathrm{i} \sqrt{\rho} \ddot{\theta} \end{aligned} \tag{6.8}$$

$$\begin{aligned} \nabla^2 \phi = \nabla \cdot \nabla \left(\sqrt{\rho} e^{\mathrm{i}\theta} \right) &= \nabla \cdot \left(\frac{1}{2\sqrt{\rho}} \nabla \rho e^{\mathrm{i}\theta} + \mathrm{i} \sqrt{\rho} e^{\mathrm{i}\theta} \nabla \theta \right) \\ &= -\frac{1}{4\rho\sqrt{\rho}} (\nabla \rho)^2 + \frac{1}{2\sqrt{\rho}} \nabla^2 \rho + \mathrm{i} \frac{1}{\sqrt{\rho}} \nabla \rho \cdot \nabla \theta - \sqrt{\rho} (\nabla \theta)^2 + \mathrm{i} \sqrt{\rho} \nabla^2 \theta \end{aligned} \tag{6.9}$$

The real part gives:

$$-\dot{\rho}^2 + 2\rho\ddot{\rho} - 4\rho^2\dot{\theta}^2 = \frac{1}{2m}\left(-(\nabla\rho)^2 + 2\rho\nabla^2\rho - 4\rho^2(\nabla\theta)^2\right) - \frac{4U}{\hbar^2}\rho^3 \tag{6.10}$$

The imaginary part gives:

$$\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = \frac{1}{2m}\left(\nabla\rho\cdot\nabla\theta + \rho\nabla^2\theta\right) \tag{6.11}$$

Considering the small amplitude oscillations of the phase and the amplitude $\rho \to \rho + \delta\rho, \theta \to \delta\theta$ and only keeping ∂_t^2 and ∇^2 term, the real part Eq. 6.10 gives the gapped Higgs mode and the imaginary part Eq. 6.11 gives the gapless Goldstone mode:

$$\begin{aligned}\ddot{\delta\rho} &= \frac{1}{2m}\nabla^2\delta\rho - \frac{2U}{\hbar^2}\rho^2 \Rightarrow \omega^2 = \frac{k^2}{2m} + 2a \\ \ddot{\delta\theta} &= \frac{1}{2m}\nabla^2\delta\theta \Rightarrow \omega^2 = \frac{k^2}{2m}\end{aligned} \tag{6.12}$$

8.5

Two-dimensional FHM without interactions and for a nonmagnetic state in square lattice $(a = 1)$ is

$$\begin{aligned}H_{\mathrm{FHM}} &= -J\sum_{\langle ij\rangle,\sigma} c_{i\sigma}^\dagger c_{j\sigma} - \mu\sum_i (n_{i\uparrow} + n_{i\downarrow}) \\ &= \sum_{\boldsymbol{k},\sigma}\left[-\mu - 2J(\cos k_x + \cos k_y)\right] c_{\boldsymbol{k},\sigma}^\dagger c_{\boldsymbol{k},\sigma}\end{aligned} \tag{6.13}$$

With filling number $\frac{N_\uparrow + N_\downarrow}{N_s}$ changing from 0 to 2, Fermi energy changes from $-\mu - 4J$ to $-\mu + 4J$ (for vacuum, Fermi eneygy is 0) and Fermi surface is the contour of energy in momentum space, see Fig. 6.1.

8.6

See 3.10.

8.8[1]

The Fermi-Hubbard model with $\mu = 0$ is

$$H_{\mathrm{FHM}} = -J\sum_{\langle ij\rangle,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + U\sum_i\left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right) - \mu\sum_i (n_{i\uparrow} + n_{i\downarrow}) \tag{6.14}$$

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[1]吐槽：太难算了...

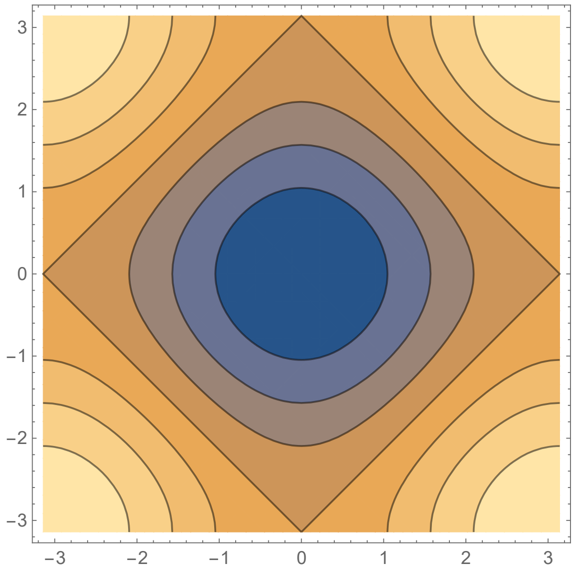

Figure 6.1: Shape of Fermi surface.

Remember $[A, BC] = \{A, B\}C - B\{A, C\}$ and $\{a_{i\rho}, a^\dagger_{j\sigma}\} = \delta_{ij}\delta_{\rho\sigma}$ for fermions. First we calculate the term

$$[l, H_{\text{FHM}}] = \left[\sum_i (-1)^{i_x+i_y} c^\dagger_{i\uparrow} c^\dagger_{i\downarrow}, H_{\text{FHM}}\right] \tag{6.15}$$

The commutator with hopping term:

$$\begin{aligned}
&[l, \sum_{\langle jk\rangle} c^\dagger_{j\uparrow} c_{k\uparrow} + c^\dagger_{j\downarrow} c_{k\downarrow}] \\
=&\sum_{i,\langle jk\rangle} \left[(-1)^{i_x+i_y} c^\dagger_{i\uparrow} c^\dagger_{i\downarrow}, c^\dagger_{j\uparrow} c_{k\uparrow} + c^\dagger_{j\downarrow} c_{k\downarrow}\right] \\
=&\sum_{i,\langle jk\rangle} \left[(-1)^{i_x+i_y} c^\dagger_{i\uparrow} c^\dagger_{i\downarrow}, c^\dagger_{j\uparrow} c_{k\uparrow}\right] + \left[(-1)^{i_x+i_y} c^\dagger_{i\uparrow} c^\dagger_{i\downarrow}, c^\dagger_{j\downarrow} c_{k\downarrow}\right] \\
=&\sum_{i,\langle jk\rangle} (-1)^{i_x+i_y} \left[c^\dagger_{i\uparrow}, c^\dagger_{j\uparrow} c_{k\uparrow}\right] c^\dagger_{i\downarrow} + (-1)^{i_x+i_y} c^\dagger_{i\uparrow} \left[c^\dagger_{i\downarrow}, c^\dagger_{j\downarrow} c_{k\downarrow}\right] \\
=&\sum_{i,\langle jk\rangle} (-1)^{i_x+i_y} \left(-c^\dagger_{j\uparrow} \delta_{ik} c^\dagger_{i\downarrow}\right) + (-1)^{i_x+i_y} \left(-c^\dagger_{i\uparrow} c^\dagger_{j\downarrow} \delta_{ik}\right) \\
=&\sum_{\langle jk\rangle} (-1)^{k_x+k_y} \left(-c^\dagger_{j\uparrow} c^\dagger_{k\downarrow} - c^\dagger_{k\uparrow} c^\dagger_{j\downarrow}\right) \\
=&0
\end{aligned} \tag{6.16}$$

The last equality is because j, k are nearest neighbors, when $k \to k+1$ the $(-1)^{kx+ky}$ term changes sign so the commutator vanishes.

The commutator with interaction term:

$$\begin{aligned}
&\left[l, \sum_j \left(n_{j\uparrow} - \frac{1}{2}\right)\left(n_{j\downarrow} - \frac{1}{2}\right)\right] \\
=&\sum_{ij}(-1)^{i_x+i_y}\left(\left[c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger, n_{j\uparrow}-1/2\right](n_{j\downarrow}-1/2) + (n_{j\uparrow}-1/2)\left[c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger, n_{j\downarrow}-1/2\right]\right) \\
=&\sum_{ij}(-1)^{i_x+i_y}\left(-c_{j\uparrow}^\dagger \delta_{ij} c_{i\downarrow}^\dagger (n_{j\downarrow}-1/2) - (n_{j\uparrow}-1/2) c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger \delta_{ij}\right) \\
=&\sum_{i}(-1)^{i_x+i_y}\left(\frac{1}{2}c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + \frac{1}{2}c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger - n_{i\uparrow} c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger\right) \\
=&\sum_{i}(-1)^{i_x+i_y}\left(c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger - (1 - c_{i\uparrow} c_{i\uparrow}^\dagger) c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger\right) \\
=&0
\end{aligned} \tag{6.17}$$

Keep in mind that $c_{i\uparrow}^\dagger c_{i\uparrow}^\dagger = c_{i\uparrow} c_{i\uparrow} = 0$.

The commutator with Zeeman term is straightforward:

$$\begin{aligned}
&\left[l, \sum_j (n_{j\uparrow} - n_{j\downarrow})\right] \\
=&\sum_i (-1)^{i_x+i_y}\left(\left[c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger, n_{i\uparrow}\right] - \left[c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger, n_{i\downarrow}\right]\right) \\
=&\sum_i (-1)^{i_x+i_y}\left(-c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger\right) \\
=&0
\end{aligned} \tag{6.18}$$

So we get $[l, H_{\mathrm{FHM}}] = 0$. Since $H_{\mathrm{FHM}} = H_{\mathrm{FHM}}^\dagger$, $[l^\dagger, H_{\mathrm{FHM}}] = [l^\dagger, H_{\mathrm{FHM}}^\dagger] = -[l, H_{\mathrm{FHM}}] = 0$.

$$\begin{aligned}
[L^x, H_{\mathrm{FHM}}] &= \left[\frac{1}{2}\left(l + l^\dagger\right), H_{\mathrm{FHM}}\right] = 0 \\
[L^y, H_{\mathrm{FHM}}] &= \left[\frac{\mathrm{i}}{2}\left(l - l^\dagger\right), H_{\mathrm{FHM}}\right] = 0
\end{aligned} \tag{6.19}$$

For $L^z = \frac{1}{2}\left(\sum_i \left(c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow}\right) - N_s\right) = \frac{1}{2}\left(\sum_i (n_{i\uparrow} + n_{i\downarrow}) - N_s\right)$, obviously it commutes with interaction term and Zeeman term in FHM Hamiltonian. We only need to show that L^z also commutes with the hopping term, which is trivial:

$$\begin{aligned}
&\sum_{i,\langle jk\rangle}\left[c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow}, c_{j\uparrow}^\dagger c_{k\uparrow} + c_{j\downarrow}^\dagger c_{k\downarrow}\right] \\
=&\sum_{i,\langle jk\rangle}\left(c_{i\uparrow}^\dagger \delta_{ij} c_{k\uparrow} - c_{j\uparrow}^\dagger \delta_{ik} c_{i\uparrow}\right) + \left(c_{i\downarrow}^\dagger \delta_{ij} c_{k\downarrow} - c_{j\downarrow}^\dagger \delta_{ik} c_{i\downarrow}\right) \\
=&\sum_{\langle jk\rangle}\left(c_{j\uparrow}^\dagger c_{k\uparrow} - c_{j\uparrow}^\dagger c_{k\uparrow}\right) + \left(c_{j\downarrow}^\dagger c_{k\downarrow} - c_{j\downarrow}^\dagger c_{k\downarrow}\right) \\
=&0
\end{aligned} \tag{6.20}$$

Finally, we proved $[\boldsymbol{L}, H_{\mathrm{FHM}}] = 0$ when $\mu = 0$.